

ESTIMATES FOR RADIAL SOLUTIONS OF THE HOMOGENEOUS LANDAU EQUATION WITH COULOMB POTENTIAL

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ABSTRACT. Motivated by the question of existence of global solutions, we obtain pointwise upper bounds for radially symmetric and monotone solutions to the homogeneous Landau equation with Coulomb potential. The estimates say that blow up in the $L^\infty(\mathbb{R}^3)$ -norm at a finite time T can occur only if the $L^{3/2}(\mathbb{R}^3)$ -norm of the solution concentrates for times close to T . The bounds are obtained using the comparison principle for the Landau equation and for the associated mass function.

This method provides long-time existence results for the isotropic version of the Landau equation with Coulomb potential, recently introduced by Krieger and Strain.

1. INTRODUCTION

This manuscript is concerned with the Cauchy problem for the homogeneous Landau equation: such equation takes the general form

$$\partial_t f(v, t) = Q(f, f), \quad f(v, 0) = f_{in}(v), \quad v \in \mathbb{R}^3, \quad t > 0, \quad (1.1)$$

where $Q(f, f)$ is a quadratic operator known as the Landau collisional operator

$$Q(f, f) = \operatorname{div} \left(\int_{\mathbb{R}^3} A(v - y) (f(y) \nabla_v f(v) - f(v) \nabla_y f(y)) dy \right). \quad (1.2)$$

The term $A(v)$ denotes a positive and symmetric matrix

$$A(v) := \frac{1}{8\pi} \left(\operatorname{Id} - \frac{v \otimes v}{|v|^2} \right) \varphi(|v|), \quad v \neq 0,$$

which acts as the projection operator onto the space orthogonal to the vector v . The function $\varphi(|v|)$ is a scalar valued function determined from the original Boltzmann kernel describing how particles interact. If the interaction strength between particles at a distance r is proportional to r^{-s} , then

$$\varphi(|v|) := |v|^{\gamma+2}, \quad \gamma = \frac{(s-5)}{(s-1)}. \quad (1.3)$$

Any solution to (1.1)-(1.2) is an integrable and nonnegative scalar field $f(v, t) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^+$. Equation (1.1) describes the evolution of a plasma in spatially homogeneous regimes, which means that the density function f depends only on the velocity component v . Landau's original intent in deriving this approximation was to make sense of the Boltzmann collisional operator, which always diverges when considering purely grazing collisions.

The Cauchy problem for (1.1)-(1.3) is very well understood for the case of hard potentials, which correspond to $\gamma \geq 0$ above. Desvillettes and Villani showed the existence of global classical solutions for hard potentials and studied its long time behavior, see [2, 3, 15] and

references therein. In this case there is a unique global smooth solution, which converges exponentially to an equilibrium distribution, known as the Maxwellian function

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

Analyzing the soft potentials case, $\gamma < 0$, has proved to be more difficult. Using a probabilistic approach, Fournier and Guerin [4] obtained uniqueness and existence of weak solutions for the case of moderately soft potentials ($\gamma \in [-2, 0]$), uniqueness was also shown to hold for bounded solutions in the Coulomb case in [5]. On the other hand, in recent work of Alexnadre, Liao and Lin [13] it is proved -for moderately soft potentials- that the L^2 of the solution remains bounded for finite times, as long as the initial data is in L^2 . For $\gamma \in [-3, -2]$, short time existence is known (going back to work of Arsenev and Peskov [1] for initial data in L^∞ , see also [13]) as well as global existence under a smallness L^2 assumption on the initial configuration, as proved by Wu [16]. In a different direction, Villani [14] introduced the so called H -solutions, which enjoy (weak) a priori bounds in a weighted Sobolev space. However, the issue of their uniqueness and regularity (i.e. no finite time break down occurs) has remained open, even for smooth initial data: see [15, Chapter 1, Chapter 5] for further discussion.

Guo in [8] employs a completely different approach based on perturbation theory for the existence of periodic solutions to the spatially inhomogeneous Landau equation in \mathbb{R}^3 . He shows that if the initial data is sufficiently close to the unique equilibrium in a certain high Sobolev norm then a unique global solution exists. Moreover, as remarked in [8], this approach also extends to the case of potentials (1.3) where γ might even take values below -3 .

Due to the lack of a global well-posedness theory, several conjectures about possible finite-time blow up for general initial data have been made throughout the years. In [15] Villani discussed the possibility that (1.1)-(1.3) could blow up for $\gamma = -3$. Note that for smooth solutions (1.1)-(1.3) with $\gamma = -3$ can be rewritten as

$$\partial_t f = \operatorname{div}(A[f]\nabla f - f\nabla a[f]) = \operatorname{Tr}(A[f]D^2 f) + f^2, \quad (1.4)$$

where

$$A[f] := A(v) * f = \frac{1}{8\pi|v|} \left(\operatorname{Id} - \frac{v \otimes v}{|v|^2} \right) * f, \quad \Delta a = -f.$$

Equation (1.4) can be thought of as a quasi-linear nonlocal heat equation. Supports for blow-up conjectures were given by the fact that (1.4) is reminiscent of the well studied semilinear heat equation

$$\partial_t f = \Delta f + f^2, \quad (1.5)$$

Blow up for (1.5) in L^∞ implies blow up of every L^p norm with $p > 3/2$, as shown for instance in work of Giga and Kohn [6], where the singularities are studied using self-similar variables.

However, despite the apparent similarities, equation (1.4) behaves differently than (1.5). The Landau equation admits a richer class of equilibrium solution: every Maxwellian \mathcal{M} solves $Q(\mathcal{M}, \mathcal{M}) = 0$ which holds, in particular, for those with arbitrarily large mass.

From a different perspective, Krieger-Strain [9] considered an isotropic version of (1.4)

$$\partial_t f = a[f]\Delta f + \alpha f^2, \quad \alpha \in [0, 1], \quad (1.6)$$

and showed global existence of smooth radial solutions starting from radial initial data when $\alpha < 2/3$. This range for α later was expanded to any $\alpha < 74/75$ by means of a non-local

inequality obtained by Gressman, Krieger and Strain [7]. Note that when $\alpha = 1$, the above equation can be written in divergence form,

$$\partial_t f = \operatorname{div}(a[f]\nabla f - f\nabla a[f]). \quad (1.7)$$

These results put in evidence how a non-linear equation with a non-local diffusivity such as (1.7) behaves drastically different (and better) than (1.5).

Our main results in this manuscript are twofold. The first one gives necessary conditions for the finite time blow up of solutions to (1.4). The second (unconditional) result says that solutions to (1.7) do not blow up at all. Both theorems deal only with radially symmetric, decreasing initial conditions.

On the initial condition f_{in} we make the following assumption: for some $p > 6$, f_{in} satisfies

$$\begin{aligned} f_{in} &\in L^1(\mathbb{R}^3) \cap L^p_{\text{weak}}(\mathbb{R}^3), \quad f_{in} \geq 0, \quad \|f_{in}\|_{L^1(\mathbb{R}^3)} =: M_{in}, \\ f_{in} &\text{ radially symmetric, monotone decreasing.} \end{aligned} \quad (1.8)$$

The main results are the following.

Theorem 1.1. *Let f_{in} be as in (1.8). Then there is a smooth radial solution $f(v, t)$ of (1.4) and of (1.7) defined in $\mathbb{R}^3 \times [0, T_0)$ for a positive (possibly infinite) T_0 . Moreover, T_0 is maximal in the sense that either $T_0 = \infty$ or the $L^{3/2}$ norm of f accumulates at $T = T_0$, in particular*

$$\lim_{t \rightarrow T_0^-} \|f(\cdot, t)\|_{L^p(B_1)} = \infty, \quad \forall p > 3/2.$$

This certainly does not yield long time existence of classical solutions to (1.4). However, the ideas used in proving Theorem 1.1 can at least guarantee long time existence for (1.7).

Theorem 1.2. *Let f_{in} be as in (1.8), then there is $f : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$, smooth for positive times, solving (1.7) and such that $f(\cdot, 0) = f_{in}$.*

We approach the analysis from the point of view of nonlinear parabolic equations. The nonlocal dependence of the coefficients on the solution prevents the equation from satisfying a comparison principle : if v_0 is a contact point of two functions f and g , i.e. $f(v_0) = g(v_0)$ and everywhere else $f(v) < g(v)$, it does not hold that $Q(f, f)(v_0) \leq Q(g, g)(v_0)$. More precisely, one cannot expect an inequality such as

$$\operatorname{Tr}(A[f]D^2 f)(v_0) \leq \operatorname{Tr}(A[g]D^2 g)(v_0).$$

In fact due to the nonlocality of A one only has $A[f](v_0) \leq A[g](v_0)$. Equality $A[f](v_0) = A[g](v_0)$ holds only when $f \equiv g$ for every $v \in \mathbb{R}^3$. In addition, also maximum principle does not hold, since at a maximum point for f we only obtain $\partial_t f \leq -f\Delta a[f]$, which does not rule out finite time blow up of the maximum of f . However, even if maximum and comparison principle fail, if one can control in some L^p space the size of $a[f]$ and the ellipticity (i.e. eigenvalues) of the matrix $A[f]$, then higher regularity for f via a bootstrapping effect can be shown. Thus, the problem of regularity estimates for (1.4) becomes a question of bounding f in some L^p norm, with $p > 1$.

A previous attempt by the authors that meant to cover a much more general situation (global existence for bounded, fast decaying initial data), was ultimately undone by a computational error. However we kept the main idea of barrier arguments to show global existence results for (1.7) and conditional existence for the Landau equation (1.4).

1.1. Outline. The rest of the paper is organized as follows. After a brief review in Section 2 on nonlinear parabolic theory that will be needed to construct local solutions to the non-linear problems, in Section 3 we outline the symmetry properties of (1.4). Section 4 deals with short time existence. In Section 5 we present a barrier argument that will allow to prove conditional non-blow up results for the Landau equation and global well-posedness for the Krieger-Strain equation in Section 6.

1.2. Notation. Universal constants will be denoted by c, c_0, c_1, C_0, C_1, C . Vectors in \mathbb{R}^3 will be denoted by v, w, x, y and so on, the inner product between v and w will be written (v, w) . $B_R(v_0)$ denotes closed ball of radius R centered at v_0 , if $v_0 = 0$ we simply write B_R . The identity matrix will be noted by \mathbb{I} , the trace of a matrix X will be denoted $\text{Tr}(X)$. The initial distribution for the Cauchy problem will always be denoted by f_{in} .

The letter Ω denotes a general compact subset of \mathbb{R}^3 . $Q \subset \mathbb{R}^3 \times \mathbb{R}_+$ is a space-time cylinder of parabolic diameter R with $R > 0$ a general constant, unless otherwise specified. $\partial_p Q$ denotes the parabolic boundary of Q .

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2. A RAPID REVIEW OF LINEAR PARABOLIC EQUATIONS

We will work with two bilinear operators, namely the one associated to the equation

$$Q_{\mathcal{L}}(g, f) := \text{div}(A[g]\nabla f - f\nabla a[g]) = \text{tr}[A[g]D^2 f] + fg,$$

and the one associated to the equation of Krieger and Strain,

$$Q_{\mathcal{KS}}(g, f) := \text{div}(a[g]\nabla f - f\nabla a[g]) = a[g]\Delta f + fg.$$

As it is well known, through $Q_{\mathcal{L}}$ (and also $Q_{\mathcal{KS}}$) any $g : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$, gives rise to a linear elliptic operator with variable coefficients, as follows:

$$\begin{aligned} \phi \rightarrow Q_{\mathcal{L}}(g, \phi) &:= \text{div}(A[g]\nabla \phi - \phi\nabla a[g]) = \text{tr}(A[g]D^2 \phi) + \phi g, \\ \phi \rightarrow Q_{\mathcal{KS}}(g, \phi) &:= \text{div}(a[g]\nabla \phi - \phi\nabla a[g]) = a[g]\Delta \phi + \phi g. \end{aligned}$$

Accordingly, given such a g and initial data f_{in} , one considers the linear Cauchy problem,

$$\begin{cases} \partial_t f &= Q(g, f) \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) &= f_{\text{in}}, \end{cases} \quad (2.1)$$

both when $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$.

Remarks 1. Note that $Q_{\mathcal{L}}(g, f)$ and $Q_{\mathcal{KS}}(g, f)$ can both be expressed as a divergence, so any solution to (2.1) preserves its mass over time, i.e. $\|f(\cdot, t)\|_{L^1(\mathbb{R}^3)} = \|f_{\text{in}}(\cdot)\|_{L^1(\mathbb{R}^3)} =: M_{\text{in}}$ for all $t > 0$.

Lemma 2.1. *Let $f_{\text{in}} : \mathbb{R}^3 \rightarrow \mathbb{R}$, $g : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be non-negative functions such that*

$$\begin{aligned} f_{\text{in}} &\in L^1(\mathbb{R}^3) \cap C^{2,\beta}(\mathbb{R}^3), \\ g &\in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3)) \cap C^\beta(\mathbb{R}^3 \times \mathbb{R}_+), \quad \text{for some } \beta \in (0, 1). \end{aligned} \quad (2.2)$$

For any $\delta > 0$, there exists a unique $f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ which is a classical solution of

$$\begin{cases} \partial_t f &= \delta \Delta f + Q(g, f) \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) &= f_{\text{in}}, \end{cases} \quad (2.3)$$

where $Q(\cdot, \cdot)$ denotes either $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$.

Next recall several parabolic regularity estimates dealing with equations of the form

$$\partial_t f = \operatorname{div} (B \nabla f + f b),$$

where $f : Q \rightarrow \mathbb{R}$ and $Q = B_R(x_0) \times (t_0 - R^2, t_0) \subset \mathbb{R}^d \times \mathbb{R}$ is the parabolic cylinder of radius R centered at some points x_0, t_0 . The first two theorems are respectively a local Hölder estimate (De Giorgi-Nash-Moser) and a L^∞ estimate for f in terms of its boundary data (Stampacchia estimate), see [10, Chapter III, Theorem 10.1, page 204] and [10, Chapter IV, Theorem 10.1, page 351] as well as [12, Chapter VI, Theorem 6.29 p. 131] for the respective proofs. The main point of these theorems is that they do not require any regularity assumption on the diffusion matrix B (beyond ellipticity and boundedness).

Theorem 2.2. (De Giorgi-Nash-Moser estimate.) Suppose f is a weak solution of the equation

$$\partial_t f = \operatorname{div} (B \nabla f + f b),$$

where b is a vector field and B is a symmetric matrix such that

$$\lambda \mathbb{I} \leq B(v, t) \leq \Lambda \mathbb{I} \text{ a.e. in } Q.$$

Then, there is some $\alpha \in (0, 1)$ and $C > 0$ such that the following estimate holds:

$$[f]_{C^{\alpha, \alpha/2}(Q_{1/2})} \leq C \left(\|f\|_{L^\infty(Q)} + \|g\|_{L^\infty(Q)} + R^2 \|b\|_{L^\infty(Q)} \right), \quad (2.4)$$

where $Q_{1/2} := B_{R/2}(x_0) \times (t_0 - (R/2)^2, t_0)$ and α and C are determined by λ, Λ, R and d .

Theorem 2.3. (Stampacchia estimate.) If f is a weak solution of

$$\partial_t f \leq \operatorname{div} (B \nabla f + b),$$

with B and b as in the previous theorem, there exists a constant $C > 0$ such that

$$\|f\|_{L^\infty(Q)} \leq C \left(\|f\|_{L^\infty(\partial_p Q)} + \|b\|_{L^d(Q)} \right), \quad (2.5)$$

as before, C is determined by λ, Λ, d and R .

We also recall the interior classical regularity estimates when the coefficients are Hölder continuous in time and space. See [10, Chapter IV] or also [12, Chapter III, p. 59] for a proof.

Theorem 2.4. (Schauder estimates.) If $B, b \in C^{\beta, \beta/2}(Q)$, then there is a finite C such that

$$[D^2 f]_{C^{\beta, \alpha/2}(Q_{1/2})} + [\partial_t f]_{C^{\beta, \beta/2}(Q_{1/2})} \leq C \left(\lambda, \Lambda, R, \|B\|_{C^{\beta, \beta/2}(Q)}, \|b\|_{C^{\beta, \beta/2}(Q)}, \|f\|_{L^\infty(Q)} \right).$$

3. RADIAL SYMMETRY

This section is devoted to some technical lemmas. The proofs of the first two propositions are rather technical and can be found in the Appendix.

Proposition 3.1. *Suppose f_{in} and $g(\cdot, t)$ are both radially symmetric, and let $Q(\cdot, \cdot)$ denote either $Q_{\mathcal{L}}$ or $Q_{\mathcal{KS}}$. Then any solution of the linear Cauchy problem*

$$\partial_t f = Q(g, f), \quad f(v, 0) = f_{\text{in}}(v),$$

is radially symmetric for all t . Furthermore, if f_{in} and g are radially decreasing, then so is f .

Let $h : \mathbb{R}^3 \rightarrow \mathbb{R}_+$, define

$$A^*[h](v) := (A[h](v)\hat{v}, \hat{v}), \quad v \neq 0, \quad \hat{v} := v|v|^{-1}. \quad (3.1)$$

There are two useful expressions for $A^*[h]$ and $a[h]$ when h is radially symmetric.

Proposition 3.2. *Let $h \in L^1(\mathbb{R}^3)$ be radially symmetric and non-negative. Then*

$$A^*[h](v) = \frac{1}{12\pi|v|^3} \int_{B_{|v|}} h(w)|w|^2 dw + \frac{1}{12\pi} \int_{B_{|v|}^c} \frac{h(w)}{|w|} dw, \quad (3.2)$$

$$a[h](v) = \frac{1}{4\pi|v|} \int_{B_{|v|}} h(w) dw + \frac{1}{4\pi} \int_{B_{|v|}^c} \frac{h(w)}{|w|} dw. \quad (3.3)$$

The second formula above is simply the classical formula for the Newtonian potential in the case of radial symmetry, the formula for $A^*[h]$ is new and

Lemma 3.3. *Let $h \in L^1(\mathbb{R}^3)$ be a non-negative, decreasing radial function.*

(1) *If*

$$\int_{B_{R_1} \setminus B_{R_0}} h dv \geq \delta > 0,$$

for some $\delta > 0$ and $0 < R_0 < R_1$ then,

$$A[h](v) \geq \frac{\delta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3} \mathbb{I}. \quad (3.4)$$

(2) *If h is bounded, i.e. if $\|h\|_{L^\infty(\mathbb{R}^3)} = h(0) < +\infty$, it holds*

$$A[h](v) \leq a[h]\mathbb{I} \leq 2 \left(\frac{\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)}}{1 + |v|} \right) \mathbb{I}. \quad (3.5)$$

Proof. (1) Let $A^*[h]$ be as in (3.2). If $|v| \geq R_1$, then

$$\begin{aligned} A^*[h](v) &\geq \frac{1}{12\pi|v|^3} \int_{B_{R_1}} h(w)|w|^2 dw \geq \frac{1}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w)|w|^2 dw \\ &\geq \frac{R_0^2}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w, t) dw \geq \frac{\delta R_0^2}{12\pi|v|^3}. \end{aligned}$$

Note that Proposition (3.2) guarantees that $A^*[h]$ is radially decreasing. Thus,

$$A^*[h](v) \geq \frac{\delta R_0^2}{12\pi R_1^3}, \quad \forall v \in B_{R_1}.$$

Combining both estimates, we conclude that

$$A^*[h](v) \geq \frac{\delta R_0^2}{12\pi(1+R_1^3)} \frac{1}{1+|v|^3}.$$

(2) If $h \in L^\infty$, we may use (3.2) to obtain the estimate

$$\begin{aligned} A[h] \leq a[h](v)\mathbb{I} &\leq \left(\frac{h(0)}{4\pi|v|} \int_{B_{|v|}} dw + \frac{1}{4\pi} \int_{B_1^c} h(w) dw + \frac{1}{4\pi} \int_{B_1} \frac{h(w)}{|w|} dw \right) \mathbb{I} \\ &\leq (\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)}) \mathbb{I}, \quad \text{if } |v| \leq 1, \end{aligned}$$

and

$$A[h] \leq a[h](v)\mathbb{I} \leq \left(\frac{\|h\|_{L^1(\mathbb{R}^3)}}{2\pi|v|} \right) \mathbb{I}, \quad \text{if } |v| \geq 1.$$

□

Proposition 3.4. *Let h be a positive and radially symmetric decreasing function. For any $\gamma \in (0, 1)$ define $U_\gamma(v)$ as*

$$U_\gamma(v) := |v|^{-\gamma}.$$

Then,

$$Q_{\mathcal{L}}(h, U_\gamma), Q_{\mathcal{KS}}(h, U_\gamma) \leq U_\gamma \left(-\frac{1}{3}\gamma(1-\gamma)a[h]|v|^{-2} + h \right).$$

Proof. As U_γ is radial

$$\nabla U_\gamma(v) = U'_\gamma(v) \frac{v}{|v|}, \quad D^2 U_\gamma(v) = U''_\gamma(v) \frac{v}{|v|} \otimes \frac{v}{|v|} + U'_\gamma(v) \frac{1}{|v|} (\mathbb{I} - \frac{v}{|v|} \otimes \frac{v}{|v|}).$$

Thus,

$$Q_{\mathcal{L}}(h, U_\gamma) = \text{tr}(A[h]D^2 U_\gamma) + hU_\gamma = A^*[h]U''_\gamma + \frac{a[h] - A^*[h]}{|v|} U'_\gamma + hU_\gamma.$$

In particular, since $U'_\gamma = -\gamma r^{-1}U_\gamma$, $U''_\gamma = \gamma(\gamma+1)|v|^{-2}U_\gamma$, it holds

$$Q_{\mathcal{L}}(h, U_\gamma) = U_\gamma \left(\gamma(\gamma+1)A^*[h]|v|^{-2} - \gamma(a[h] - A^*[h])|v|^{-2} + h \right).$$

The thesis follows by noticing that $A^*[h] \leq \frac{1}{3}a[h]$.

As for $Q_{\mathcal{KS}}(h, U_\gamma)$, straightforward computations show that

$$Q_{\mathcal{KS}}(h, U_\gamma) = U_\gamma \left(-\gamma(1-\gamma)a[h]|v|^{-2} + h \right).$$

□

4. SHORT TIME EXISTENCE.

In the following section, the operator Q denotes either $Q_{\mathcal{L}}$ or $Q_{\mathcal{KS}}$, unless otherwise specified. A sequence $\{f_k^{(\delta)}\}_k$ of approximate solutions to the non-linear Cauchy problem for Q is built iteratively as follows. First set $f_0^{(\delta)}(v, t) := f_{\text{in}}(v)$ for all v and $t > 0$. Then, for f_{in} as in (2.2) and (1.8), $k \in \mathbb{N}$ let $f_k^{(\delta)}$ be the unique classical solution of

$$\begin{cases} \partial_t f &= \delta \Delta f + Q(f_{k-1}^{(\delta)}, f) \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) &= f_{\text{in}}, \end{cases} \quad (4.1)$$

which exists thanks to Lemma 2.1. Moreover, each $f_k^{(\delta)}$ is radially symmetric and monotone, by Proposition 3.1 and the assumptions (1.8) for f_{in} .

By virtue of $f_k^{(\delta)}$ being smooth and monotone decaying, it follows that $f_k^{(\delta)}(0, t) \geq f_k^{(\delta)}(v, t)$ and $D^2 f^{(\delta)}(0, t) \leq 0$. Thus, $f_k^{(\delta)}(0, t)$, seen solely as a function of t , always satisfies the differential inequality

$$\partial_t f_k^{(\delta)}(0, t) \leq f_{k-1}^{(\delta)}(0, t) f_k^{(\delta)}(0, t), \quad k \geq 1. \quad (4.2)$$

We next prove that $\|f_k^{(\delta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded in δ, k for $t \in [0, T_0]$, where T_0 depends only on f_{in} .

Lemma 4.1. *Let $f_k^{(\delta)}(v, t)$ be given by (4.1). Then,*

$$f_k^{(\delta)}(0, t) \leq \frac{f_{\text{in}}(0)}{1 - f_{\text{in}}(0)t}, \quad \forall t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right). \quad (4.3)$$

Proof. Define functions $h_k(t)$ iteratively for $k \in \mathbb{N}$ by

$$h'_k(t) = h_{k-1}(t)h_k(t), \quad h_k(0) = f_{\text{in}}(0), \quad h_0 \equiv f_{\text{in}}(0).$$

First we observe that $h_k \leq h_{k+1}$ for all k ; hence $h'_k \leq h_k^2$ for all $k \geq 0$. The thesis follows since

$$f_k^{(\delta)}(0, t) \leq h_k(t) \leq \frac{f_{\text{in}}(0)}{1 - f_{\text{in}}(0)t}.$$

□

Proposition 4.2. *Let $\{f_k\}$ be given by (4.1) and suppose $T > 0$ is such that*

$$C(T) := \limsup_k \|f_k^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, T])} < \infty$$

Then, there are $r = r(T, C(T), f_{\text{in}})$ and $R = R(T, C(T), f_{\text{in}})$, with $r < R$ and such that

$$\int_{B_R \setminus B_r} f_k^{(\delta)}(v, t) dv \geq \frac{1}{2} \quad \forall k \geq 1 \quad t \in [0, T]. \quad (4.4)$$

Proof. First, let us compute the rate of change for the second moment of f , using the evolution equation for f :

$$\frac{d}{dt} \int_{\mathbb{R}^3} f_k^{(\delta)}(v, t) |v|^2 dv = \int_{\mathbb{R}^3} \left(\delta \Delta f_k^{(\delta)} + Q(f_{k-1}^{(\delta)}, f_k^{(\delta)}) \right) |v|^2 dv.$$

For simplicity, $B[g]$ will denote either the matrix $A[g]$ for $Q = Q_{\mathcal{L}}$ or the matrix $a[g]\mathbb{I}$ for $Q = Q_{\mathcal{KS}}$. Integration by parts above yields

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} f_k^{(\delta)}(v, t) |v|^2 dv &= -2 \int_{\mathbb{R}^3} (B[f_{k-1}^{(\delta)}] \nabla f_k^{(\delta)} - f_k^{(\delta)} \nabla a[f_{k-1}^{(\delta)}], v) dv \\ &= -2 \int_{\mathbb{R}^3} (\nabla f_k^{(\delta)}, B[f_{k-1}^{(\delta)}] v) dv + 2 \int_{\mathbb{R}^3} f_k^{(\delta)} (\nabla a[f_{k-1}^{(\delta)}], v) dv \\ &= 2 \int_{\mathbb{R}^3} f_k^{(\delta)} (\text{div}(B[f_{k-1}^{(\delta)}] v) + (\nabla a[f_{k-1}^{(\delta)}], v)) dv. \end{aligned}$$

Note that $\text{div}(B[f_{k-1}^{(\delta)}] v) + (\nabla a[f_{k-1}^{(\delta)}], v) = \text{tr}[B[f_{k-1}^{(\delta)}]\mathbb{I}] + 2(\nabla a[f_{k-1}^{(\delta)}], v)$, whether $B[\cdot]$ is given by $A[\cdot]$ or $a[\cdot]$. Moreover,

$$\text{tr}(B[f_{k-1}^{(\delta)}]) \leq 3a[f_{k-1}^{(\delta)}],$$

and

$$\nabla a[f_{k-1}^{(\delta)}](v, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} f_{k-1}^{(\delta)}(w, t) \frac{(v-w)}{|v-w|^3} dw.$$

Hence

$$\begin{aligned} |\nabla a[f_{k-1}^{(\delta)}](v, t)| &\leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f_{k-1}^{(\delta)}(w, t)}{|v-w|^2} dw \\ &= \frac{1}{4\pi} \int_{B_1(v)} \frac{f_{k-1}^{(\delta)}(w, t)}{|v-w|^2} dw + \frac{1}{4\pi} \int_{B_1(v)^c} \frac{f_{k-1}^{(\delta)}(w, t)}{|v-w|^2} dw \\ &\leq \frac{1}{4\pi} 4\pi \|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{4\pi} \|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^1(\mathbb{R}^3)}. \end{aligned}$$

Therefore, for $t < T$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} f_k^{(\delta)} |v|^2 dv &\leq 16 \|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^1(\mathbb{R}^3)} \left(\|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} + \|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^1(\mathbb{R}^3)} \right) \\ &\quad + 4 \left(\|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} + \|f_{k-1}^{(\delta)}(\cdot, t)\|_{L^1(\mathbb{R}^3)} \right) \int_{\mathbb{R}^3} f_k^{(\delta)} |v|^2 dv. \end{aligned}$$

which implies,

$$\int_{\mathbb{R}^3} f_k^{(\delta)}(v, t) |v|^2 dv \leq e^{4T(C(T)+M_{in})} \left(\int_{\mathbb{R}^3} f_{in} |v|^2 dv + 4 \right). \quad (4.5)$$

On the other hand, for any r, R with $R > r > 0$ there is the obvious lower bound,

$$\begin{aligned} \int_{B_R \setminus B_r} f_k^{(\delta)}(v, t) dv &= 1 - \int_{B_R^c} f_k^{(\delta)}(v, t) dv - \int_{B_r} f_k^{(\delta)}(v, t) dv \\ &\geq 1 - R^{-2} \int_{\mathbb{R}^3} f_k^{(\delta)}(v, t) |v|^2 dv - \frac{4}{3} \pi r^3 C(T). \end{aligned}$$

The thesis follows by choosing

$$\begin{aligned} R &:= \sqrt{6} \left(\int_{\mathbb{R}^3} f_{in} |v|^2 dv + 4 \right)^{1/2} e^{2T(C(T)+M_{in})}, \\ r &:= (4\pi C(T))^{-1/3}, \end{aligned}$$

since

$$1 - R^{-2} \int_{\mathbb{R}^3} f_k^{(\delta)}(v, t) |v|^2 dv - \frac{4}{3} \pi r^3 C(T) \geq \frac{1}{2}.$$

□

Proposition 4.3. *Let f_k^δ be the unique solution to (4.1), and $M_{in} := \int_{\mathbb{R}^3} f_{in}(v) dv$. Then*

(1) *There exists a constant $c_0 = c_0(f_{in}, T)$ such that*

$$A^*[f_k^\delta](v) \geq \frac{c_0}{1+|v|^3} \mathbb{I}, \quad \forall t \in [0, T], \quad \forall k \geq 1. \quad (4.6)$$

(2) *$A[f_k]$ and $a[f_k]$ satisfy the pointwise bound*

$$A[f_k^\delta](v, t) \leq a[f_k^\delta](v, t) \leq \frac{2(\|f_k^\delta\|_{L^\infty(\mathbb{R}^3 \times [0, T])} + M_{in})}{1+|v|} \mathbb{I}, \quad \forall v \in \mathbb{R}^3. \quad (4.7)$$

(3) $A[f_k^\delta](\cdot, t) \in C^{1,\alpha}(\mathbb{R}^3)$ for every $\alpha \in [0, 1]$ with a bound of the form,

$$\|\nabla A[f_k^\delta]\|_{C^\alpha(\mathbb{R}^3)} \leq c(M_{in}, \|f_k^\delta\|_{L^\infty(\mathbb{R}^3 \times [0, T])}).$$

(4) $A[f_k^\delta]$ is locally Hölder continuous in time, and

$$\|A[f_k^\delta](v, \cdot)\|_{C^\beta[0, T]} \leq c(f_{in}, R, \|f_k^\delta\|_{L^\infty(\mathbb{R}^3 \times [0, T])}), \quad \forall v \in B_R.$$

Proof. The proof of (1) and (2) follows directly from Proposition 4.2 and Lemma 3.3. For (3) first notice that $a[f_k^\delta] \in W^{2,p}(\mathbb{R}^3)$ for any $1 \leq p \leq +\infty$ since $f_k^\delta \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. In particular

$$\|a[f_k^\delta]\|_{W^{2,p}} \leq C\|f_k^\delta\|_{L^p} \leq C\|f_k^\delta\|_{L^\infty(\mathbb{R}^3)}^{p-1}\|f_k^\delta\|_{L^1(\mathbb{R}^3)},$$

where C is a dimensional constant. Hence Morrey's inequality implies $a[f_k^\delta] \in C^{1,\alpha}(\mathbb{R}^3)$ for any $0 < \alpha < 1$. We use now the fact that $A[f_k^\delta]$ can be rewritten as

$$A[f_k^\delta] = D^2(\Delta^{-2}f_k^\delta),$$

with D^2 being the Hessian matrix, an apply $W^{2,p}$ Calderon-Zygmund estimates to the kernels $\partial_{i,j}(\Delta^{-2}f)$ we get we get:

$$\|D^2 A[f_k^\delta]\|_{L^p(\mathbb{R}^3)} \leq A_p \|a[f_k^\delta]\|_{W^{2,p}},$$

with A_p a dimensional constant.

To prove (4) consider any function $h : \mathbb{R}^3 \rightarrow \mathbb{R}$: it holds

$$\begin{aligned} |A[h](v_0)| &\leq |a[h](v_0)| \leq \frac{1}{8\pi} \int_{B_R(v_0)} \frac{h(v)}{|v_0 - v|} dv + \frac{1}{8\pi} \int_{B_R(v_0)^c} \frac{h(v)}{|v_0 - v|} dv \\ &\leq \frac{1}{8\pi} \|h\|_{L^\infty(B_R(v_0))} \frac{4\pi}{2} R^2 + \frac{1}{8\pi R} \|h\|_{L^1(B_R(v_0)^c)} \\ &\leq \frac{R^2}{4} \|h\|_{L^\infty(B_R(v_0))} + \frac{1}{8\pi R} \|h\|_{L^1(B_R^c(v_0))}. \end{aligned}$$

The above estimate for $h = f_k^\delta(\cdot, t) - f_k^\delta(\cdot, s)$ ($t, s \in [0, T]$) yields

$$|A[f_k^\delta](v_0, t) - A[f_k^\delta](v_0, s)| \leq \frac{R^2}{4} \|f_k^\delta(\cdot, t) - f_k^\delta(\cdot, s)\|_{L^\infty(B_R(v_0))} + \frac{1}{8\pi R} \|f_k^\delta(\cdot, t) - f_k^\delta(\cdot, s)\|_{L^1(B_R^c(v_0))}.$$

Since f_k^δ is locally Hölder continuous in time, (4) follows. \square

Proposition 4.4. Let $w_k := f_k^{(\delta)} - f_{k-1}^{(\delta)}$ and

$$T_1 := \sup\{t \geq 0 \mid \limsup_k \|f_k^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, t])} < \infty\}.$$

If $0 < T < T_1$, then

$$\|w_k\|_{L^\infty([0, T], L^\infty \cap L^1(\mathbb{R}^3))} \leq \frac{1}{2} \|w_{k-1}\|_{L^\infty([0, T], L^\infty \cap L^1(\mathbb{R}^3))}. \quad (4.8)$$

Proof. Step 1. The function w_k solves

$$\begin{cases} \partial_t w_k &= \delta \Delta w_k + Q(f_{k-1}^{(\delta)}, w_k) + F, \\ w_k(\cdot, 0) &= 0, \end{cases}$$

where

$$F(w_{k-1}, f_{k-1}^{(\delta)}) := Q(w_{k-1}, f_{k-1}^{(\delta)}).$$

The bound of the second moment (4.5) for $f_k^{(\delta)}$ implies that

$$\|f_k^{(\delta)}\|_{L^\infty(B_1(v) \times [0, T])} \leq C(T)(1 + |v|^5)^{-1}, \quad \forall v.$$

Theorems 2.2 and 2.4 imply that there exists a positive constant $\tilde{C} = \tilde{C}(M_{in}, C(T))$ such that

$$\|D^2 f_k^{(\delta)}\|_{L^\infty(B_1(v) \times [0, T])} \leq \tilde{C}(1 + |v|^5)^{-1}, \quad \forall v. \quad (4.9)$$

Moreover from

$$\begin{aligned} \|A[h(\cdot, t)]\|_{L^\infty([0, T] \times \mathbb{R}^3)} &\leq \|a[h]\|_{L^\infty([0, T] \times \mathbb{R}^3)} \mathbb{I} \\ &\leq (\|h(\cdot, t)\|_{L^\infty([0, T] \times \mathbb{R}^3)} + \|h(\cdot, t)\|_{L^1([0, T] \times \mathbb{R}^3)}) \mathbb{I}, \end{aligned} \quad (4.10)$$

it follows that, for $v \in \mathbb{R}^3$, $t \in (0, T_0)$,

$$|F(v, t)| \leq \tilde{C} \left(\frac{\|w_{k-1}(\cdot, t)\|_{L^\infty([0, T] \times \mathbb{R}^3)} + \|w_{k-1}(\cdot, t)\|_{L^1([0, T] \times \mathbb{R}^3)}}{(1 + |v|^5)} \right).$$

Step 2. Let $h(v) := (1 + |v|^2)^{-2}$, then

$$D^2 h = -4(1 + |v|^2)^{-3} \mathbb{I} + 24(1 + |v|^2)^{-4} v \otimes v.$$

So,

$$\begin{aligned} |\text{tr}((\delta \mathbb{I} + A[f_{k-1}^{(\delta)}])D^2 h)| &\leq \frac{12\delta}{(1 + |v|^2)^3} \left(-1 + \frac{2|v|^2}{(1 + |v|^2)} \right) \\ &\quad - 4 \frac{a[f_{k-1}^{(\delta)}]}{(1 + |v|^2)^3} + \frac{24}{(1 + |v|^2)^4} (A[f_{k-1}^{(\delta)}]v, v) \\ &\leq \frac{12\delta}{(1 + |v|^2)^3} - 4 \frac{a[f_{k-1}^{(\delta)}]}{(1 + |v|^2)^3} + 8 \frac{a[f_{k-1}^{(\delta)}]|v|^2}{(1 + |v|^2)^4} \\ &\leq \frac{12\delta}{(1 + |v|^2)^3} + 4 \frac{a[f_{k-1}^{(\delta)}]}{(1 + |v|^2)^3} \left[-1 + \frac{2|v|^2}{(1 + |v|^2)} \right] \\ &\leq 12\delta h + 4 \frac{a[f_{k-1}^{(\delta)}]}{(1 + |v|^2)^3} \leq 12\delta h + 4h(C(t) + 1), \end{aligned}$$

taking into account (4.10) and the fact that $(A[f_{k-1}^{(\delta)}]v, v) = A^*[f_{k-1}^{(\delta)}](v)|v|^2$, $\text{tr} A^*[f_{k-1}^{(\delta)}] = a[f_{k-1}^{(\delta)}]$ and $A^*[f_{k-1}^{(\delta)}](v) \leq \frac{1}{3}a[f_{k-1}^{(\delta)}]$. Hence for $\delta < 1$ and $C(T) > 1$ it holds

$$\begin{aligned} |\delta \Delta h + Q(f_{k-1}^{(\delta)}, h)| &= |\text{tr}((\delta \mathbb{I} + A[f_{k-1}^{(\delta)}])D^2 h) + f_{k-1}^{(\delta)} h| \\ &\leq 12\delta h + 4h + 5hC(t) \\ &\leq 6C(T)h. \end{aligned}$$

Next, let $H(v, t) := A^{-1}(e^{At} - 1)Bh(v)$, for $A, B > 0$ to be determined; a straightforward computation shows that

$$\partial_t H = AH + Bh.$$

Hence

$$\begin{aligned} \delta \Delta H + Q(f_{k-1}^{(\delta)}, H) + F &\leq 6C(T)H \\ &\quad + \tilde{C}h \left(\|w_{k-1}(\cdot, t)\|_{L^\infty([0, T] \times \mathbb{R}^3)} + \|w_{k-1}(\cdot, t)\|_{L^1([0, T] \times \mathbb{R}^3)} \right). \end{aligned}$$

By taking

$$A := 6C(T), \quad B := 2\tilde{C}(\|w_{k-1}(\cdot, t)\|_{L^\infty([0,T] \times \mathbb{R}^3)} + \|w_{k-1}(\cdot, t)\|_{L^1([0,T] \times \mathbb{R}^3)}),$$

it follows that

$$\partial_t H \geq \delta \Delta H + Q(f_{k-1}^{(\delta)}, H) + F.$$

This means that H is a supersolution for the same parabolic equation solved by w_k . Moreover, $H(\cdot, 0) = w_k(\cdot, 0) = 0$. Then, the comparison principle implies that

$$w_k \leq H \text{ in } \mathbb{R}^3 \times \mathbb{R}_+.$$

If $t \in (0, 1)$, then for all $1 \leq p \leq +\infty$ it holds

$$\|w_k(\cdot, t)\|_{L^p(\mathbb{R}^3)} \leq \|H(\cdot, t)\|_{L^p(\mathbb{R}^3)} = BA^{-1}(e^{At} - 1)\|h\|_{L^p(\mathbb{R}^3)} \leq 2e^A Bt.$$

Hence for $T_0 < 1$ such that

$$T_0 \leq \frac{1}{16e^{6C(T)}\tilde{C}},$$

we have the following inequality:

$$\|w_k\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])} + \sup_{t \in [0, T_0]} \|w_k(\cdot, t)\|_{L^1(\mathbb{R}^3)} \leq \frac{1}{2} \left(\|w_{k-1}\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])} + \sup_{t \in [0, T_0]} \|w_{k-1}(\cdot, t)\|_{L^1(\mathbb{R}^3)} \right).$$

At time T_0 we shall restart the process that will lead to the proof of estimate (4.8) in the time interval $[T_0, 2T_0]$. Hence we can iterate the process and cover the whole time interval $[0, T]$. \square

Lemma 4.5. *Let $\{f_k^{(\delta)}\}$ be the sequence defined via (4.1) with either $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$. Let*

$$T_1 := \sup\{t \geq 0 \mid \limsup_k \|f_k^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, t])} < \infty\}.$$

For $0 < T < T_1$ the sequence of functions $f_k^{(\delta)}$ converges in $C_{\text{loc}}^{2;1}(\mathbb{R}^3 \times [0, T_0])$ to a function $f^{(\delta)}$ that solves

$$\begin{cases} \partial_t f^{(\delta)} = \delta \Delta f^{(\delta)} + Q(f^{(\delta)}, f^{(\delta)}) & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) = f_{\text{in}}. \end{cases}$$

Moreover, $T_1 = \infty$ or

$$\lim_{t \rightarrow T_1^-} \|f^{(\delta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} = \infty.$$

In either case we have the estimate $\frac{1}{2f_{\text{in}}(0)} < T_1$.

Proof. Let $T < T_1$. Proposition 4.4 implies that the operator $S : f_{k-1}^\delta \rightarrow f_k^\delta$ has a fixed point f^δ in the space $L^\infty([0, T], L^\infty \cap L^1(\mathbb{R}^3))$, such that

$$\|f^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, T])} \leq C(T).$$

In particular, the functions $f_k^{(\delta)}$ stay uniformly bounded in compact subsets of $\mathbb{R}^3 \times [0, T_1)$. By the regularity estimates in Theorem 2.2, it follows that $D^2 f_k^{(\delta)}$ and $\partial_t f_k^{(\delta)}$ are also uniformly in k and δ Hölder continuous in compact subsets of $\mathbb{R}^3 \times [0, T_1)$.

Hence $f_k^{(\delta)}$ converges uniformly in compact subsets of $\mathbb{R}^3 \times [0, T_1)$ (together with $\partial_t f_k^{(\delta)}$, $\nabla f_k^{(\delta)}$, $D^2 f_k^{(\delta)}$) to some function $f^{(\delta)}$ (and its corresponding derivatives). Therefore,

$$\lim_{j \rightarrow \infty} \partial_t f_{k_j}^{(\delta)} = \partial_t f^{(\delta)}, \quad \lim_{j \rightarrow \infty} Q(f_{k_j-1}^{(\delta)}, f_{k_j}^{(\delta)}) = Q(f^{(\delta)}, f^{(\delta)}),$$

and follows that $f^{(\delta)}$ is a solution of the initial value problem

$$\partial_t f^{(\delta)} = \delta \Delta f^{(\delta)} + Q(f^{(\delta)}, f^{(\delta)}) \text{ in } \mathbb{R}^3 \times [0, T_1), \quad f^{(\delta)}(\cdot, 0) = f_{\text{in}}.$$

That proves the first assertion of the theorem.

As for the second one, suppose that

$$\limsup_{t \rightarrow T_1^-} \|f^{(\delta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C,$$

for some finite C . This implies that there exists $0 < \rho \leq \frac{1}{4C}$ such that

$$f^{(\delta)}(0, T_1 - \rho) \leq 2C.$$

Hence one can consider the linear problem

$$\partial_t f = \delta \Delta f + Q(f_{k-1}^{(\delta)}, f), \quad x \in \mathbb{R}^3, \quad t > T_1 - \rho,$$

subject to initial conditions $f_k^\delta(\cdot, T_1 - \rho) = f^{(\delta)}(\cdot, T_1 - \rho)$. From Lemma 4.1 we know that

$$f_k^\delta(0, t) \leq \frac{f^{(\delta)}(0, T_1 - \rho)}{1 - f^{(\delta)}(0, T_1 - \rho)(t - T_1 + \rho)},$$

which implies $f_k^\delta(0, t)$ is bounded (among others) at the time $\bar{t} = \frac{1}{2f^{(\delta)}(0, T_1 - \rho)} + \varepsilon$, for some small $\varepsilon > 0$. The fact that $f^{(\delta)}(0, T_1 - \rho) \leq 2C$ and $\rho \leq \frac{1}{4C}$ implies

$$\bar{t} \geq \frac{1}{4C} + T_1 - \rho + \varepsilon \geq T_1 + \varepsilon,$$

which contradicts the definition of T_1 , unless $T_1 = +\infty$. □

Theorem 4.6. *Let $\{f^{(\delta)}\}_{\delta>0}$ be as given by Lemma 4.5, with either $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$. There exists a strictly positive and possibly infinite T_0 , such that*

$$(\text{along a subsequence}) \quad f^{(\delta)} \rightarrow f \text{ in } C_{\text{loc}}^{2;1}(\mathbb{R}^3 \times [0, T_0)),$$

where $f(v, 0) = f_{\text{in}}(v)$ and f solves (1.4) or (1.7), depending on Q . Moreover, $T_0 = \infty$ or

$$\lim_{t \rightarrow T_0^-} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} = \infty,$$

and in either case we have the estimate $\frac{1}{2f_{\text{in}}(0)} < T_0$.

Proof. Let

$$T_0 := \{t \geq 0 \mid \limsup_{\delta} \|f^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, t])} < \infty\}.$$

Clearly $T_0 > \frac{1}{2f_{\text{in}}(0)}$ and possibly $T_0 = +\infty$. In case $T_0 < \infty$

$$\begin{cases} \limsup_{\delta \rightarrow 0^+} \|f^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, t])} = +\infty, & \forall t > T_0, \\ \limsup_{\delta \rightarrow 0^+} \|f^{(\delta)}\|_{L^\infty(\mathbb{R}^3 \times [0, t])} < +\infty, & \forall t < T_0. \end{cases}$$

The local uniform convergence of the $f^{(\delta)}$ and its derivatives also guarantees that

$$\partial_t f^{(\delta)} \rightarrow \partial_t f, \quad Q(f^{(\delta)}, f^{(\delta)}) \rightarrow Q(f, f),$$

uniformly in compact subsets of $\mathbb{R}^3 \times [0, T_0)$. In conclusion,

$$\partial_t f = Q(f, f) \text{ in } \mathbb{R}^3 \times (0, T_0), \quad f(\cdot, 0) = f_{\text{in}}.$$

As for the second one, the proof mimics the one in Lemma 4.5. Suppose that

$$\limsup_{t \rightarrow T_0^-} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C < +\infty,$$

for some finite C . This implies that there exists $0 < \rho \leq \frac{1}{4C}$ such that

$$f(0, T_0 - \rho) \leq 2C.$$

Hence one can consider the problem

$$\partial_t f = \delta \Delta f + Q(f, f), \quad x \in \mathbb{R}^3, \quad t > T_0 - \rho,$$

subject to initial conditions $f^\delta(\cdot, T_0 - \rho) = f(\cdot, T_0 - \rho)$. A similar argument as in Lemma 4.1 implies that

$$f^\delta(0, t) \leq \frac{f(0, T_0 - \rho)}{1 - f(0, T_0 - \rho)(t - T_0 + \rho)},$$

which implies $f^\delta(0, t)$ is bounded (among others) at the time $\bar{t} = \frac{1}{2f(0, T_0 - \rho)} + \varepsilon$, for some small $\varepsilon > 0$. The fact that $f(0, T_0 - \rho) \leq 2C$ and $\rho \leq \frac{1}{4C}$ implies

$$\bar{t} \geq \frac{1}{4C} + T_0 - \delta + \varepsilon \geq T_0 + \varepsilon,$$

which contradicts the definition of T_0 , unless $T_0 = +\infty$. □

Remarks 2. Since all the above estimates solely depend on the $L^\infty(\mathbb{R}^3)$ - and $L^1(\mathbb{R}^3)$ -norm of the initial data, short time existence for (1.4) and (1.7) for initial data as in (1.8) can be obtained from Theorem 4.6 via a simple limit argument.

5. POINTWISE BOUNDS AND PROOF OF THEOREM 1.1

5.1. Conditional pointwise bound. The following lemma is the key argument for the proofs of Theorem 1.1 and Theorem 1.2. It consists of a barrier argument (this is where the radial symmetry and monotonicity is needed) and affords control of certain spatial L^p -norms of the solution.

We first recall that any solution to equation (1.6) or (1.4) preserves its mass over time. Moreover any solution to (1.4) preserves its energy over time, i.e. $\int_{\mathbb{R}^3} |v|^2 f(v, t) dv = \int_{\mathbb{R}^3} |v|^2 f_{\text{in}}(v) dv =: E_{\text{in}}$.

Lemma 5.1. *Suppose $f : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}_+$ is a classical solution of (2.1). Suppose there exists a modulus of continuity $\omega(r)$, some $R_0 > 0$ and $\delta > 0$ such that*

$$\sup_{t \in [0, T]} \left(\int_{B_r} g(w, t)^{3/2} dw \right)^{2/3} \leq \omega(r), \quad \forall 0 < r \leq R_0, \quad (5.1)$$

$$a[g] \geq \delta, \quad \forall 0 < |v| \leq R_0, \quad t \in [0, T]. \quad (5.2)$$

Then, for any $0 < \gamma < 1$, there is a $r_0 = r_0(\delta, \omega(\cdot), \gamma)$, $0 < r_0 < R_0$ such that

$$f(v, t) \leq \max \left\{ \frac{3}{4\pi} r_0^{\gamma-3} \|f\|_{L^1(\mathbb{R}^3)}, \left(\frac{3}{4\pi}\right)^{\gamma/3} \|f_{\text{in}}\|_{L^{3/\gamma}_{\text{weak}}} \right\} |v|^{-\gamma}, \quad \text{in } B_{r_0} \times [0, T]. \quad (5.3)$$

Remark 5.2. It is easy to see that for any radially decreasing function $h(v)$ the condition that h belongs to $L^p_w(\mathbb{R}^3)$ implies that h lies below a power function of the form $1/|v|^{3/p}$, and viceversa. More precisely,

$$\|h(v)\|_{L^p_{\text{weak}}} \leq C \Leftrightarrow h(v) \leq C \left(\frac{3}{4\pi}\right)^{1/p} |v|^{-3/p}. \quad (5.4)$$

Proof. Let U_γ be as in Lemma 3.4. We first show the existence of some $r_0 > 0$ such that

$$Q(g, U_\gamma) \leq 0 \text{ in } B_{r_0} \times [0, T]. \quad (5.5)$$

According to (5.2),

$$a[g]|v|^{-2} \geq \delta|v|^{-2}, \quad \forall |v| \in [0, R_0], \quad t \in [0, T]. \quad (5.6)$$

On the other hand, since g is radially decreasing

$$g(v, t) \leq \frac{3}{4\pi|v|^3} \int_{B_{|v|}} g(w, t) dw.$$

For all $|v| > 0$, $t \in [0, T]$ Hölder's inequality and (5.1) yield

$$\begin{aligned} g(v, t) &\leq \frac{3}{4\pi|v|^3} \left(\int_{B_{|v|}} g(w, t)^{3/2} dw \right)^{2/3} \left(\frac{4\pi}{3} |v|^3 \right)^{1/3} \\ &= \left(\frac{3}{4\pi} \right)^{2/3} \left(\int_{B_{|v|}} g(w, t)^{3/2} dw \right)^{2/3} \frac{1}{|v|^2} \leq \left(\frac{3}{4\pi} \right)^{2/3} \omega(|v|) |v|^{-2}. \end{aligned}$$

Therefore in B_{r_0} it holds,

$$\begin{aligned} Q_{\mathcal{L}}(g, U_\gamma), \quad Q_{\mathcal{KS}}(g, U_\gamma) &\leq U_\gamma \left(-\frac{1}{3}\gamma(1-\gamma)a[g]|v|^{-2} + g \right) \\ &\leq U_\gamma |v|^{-2} \left(-\frac{1}{3}\gamma(1-\gamma)\delta + \left(\frac{3}{4\pi} \right)^{2/3} \omega(r_0) \right). \end{aligned}$$

Hence (5.5) holds by choosing $\gamma \in (0, 1)$ and $r_0 \in (0, R_0]$ small enough so that

$$\left(\frac{3}{4\pi} \right)^{2/3} \omega(r_0) \leq \frac{1}{3}\gamma(1-\gamma)\delta.$$

Since $f(v, t)$ is also radially decreasing, it holds

$$f(r, t) \leq \frac{3}{4\pi|v|^3} \|f\|_{L^1(\mathbb{R}^3)}. \quad (5.7)$$

The function $\tilde{U}_\gamma(v)$ defined as

$$\tilde{U}_\gamma(v) := \max \left\{ \frac{3}{4\pi} r_0^{\gamma-3} \|f\|_{L^1(\mathbb{R}^3)}, \left(\frac{3}{4\pi}\right)^{\gamma/3} \|f_{\text{in}}\|_{L^{3/\gamma}_{\text{weak}}} \right\} |v|^{-\gamma},$$

is a supersolution for the equation solved by f in $B_{r_0} \times [0, T]$, namely

$$\begin{cases} Q(g, \tilde{U}_\gamma) \leq 0 & \text{in } B_{r_0} \times [0, T], \\ \tilde{U}_\gamma \geq f & \text{on } \partial B_{r_0} \times [0, T], \\ \tilde{U}_\gamma \geq f & \text{on } B_{r_0} \times \{0\}. \end{cases}$$

By the comparison principle, (5.4) and (5.7) it follows that $f \leq \tilde{U}_\gamma$ in $B_{r_0} \times [0, T]$. \square

The next lemma shows that the mass of any radial symmetric solution to (1.4) or (1.7) in a compact set can be controlled from below by a constant that only depends on the initial data. More precisely:

Lemma 5.3. *For f solving (1.4), there exists a constant $R > 0$ that only depends on M_{in} and E_{in} such that*

$$\int_{B_R} f(v, t) dv \geq M_{in}/2, \quad t > 0. \quad (5.8)$$

For f solving (1.7), and any radii $R > r > 0$ there are $\beta > 0$ and $C_0 > 0$ such that

$$\int_{B_R \setminus B_r} f(v, t) dv \geq C_0 e^{-\beta t} \int_{B_{4R} \setminus B_{r/4}} f_{in}(v) dv \quad t > 0. \quad (5.9)$$

Proof. If f solves (1.4), then

$$\int_{B_R(0)^c} f(v, t) dv \leq R^{-2} \int_{B_R(0)^c} f(v, t) |v|^2 dv \leq R^{-2} E_{in}.$$

Thus

$$\int_{B_R(0)} f(v, t) dv = M(f_{in}) - \int_{B_R(0)^c} f(v, t) dv \geq M_{in} - R^{-2} E_{in}.$$

Estimate (5.8) follows by choosing R big enough. The corresponding estimate (5.9) for f solving (1.7) can be found in the Appendix. \square

The next lemma says that any solution f to (1.4) or (1.7) is a bounded function for all times provided f satisfies (5.1).

Lemma 5.4. *(From $L^{3/2+}$ to L^∞ .) Let $f : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}$ be a radially symmetric, radially decreasing solution to (1.8), (1.4) and (1.7), and such that for any modulus of continuity $\omega(r)$ the following estimate holds:*

$$\sup_{t \in [0, T]} \left(\int_{B_r} f(w, t)^{3/2} dw \right)^{2/3} \leq \omega(r), \quad \forall 0 < r \leq R_0. \quad (5.10)$$

Then there exists a constant $C_0 > 0$ that only depends on f_{in} such that

$$\sup_{t \in [0, T]} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} < C_0. \quad (5.11)$$

Proof. Proposition 4.3 yields the inequality,

$$A[f](v, t) \geq \lambda \mathbb{I}, \quad \forall (v, t) \in Q,$$

where $\lambda = \lambda(f_{in}, T)$. On the other hand, since f is radially decreasing,

$$f(v, t) \leq \frac{3}{4\pi} \|f\|_{L^1} r_0^{-3} \leq \frac{3}{4\pi} \|f_{in}\|_{L^1} r_0^{-3}, \quad \forall v \notin B_r.$$

Thus,

$$f(v, t) \leq \frac{3}{4\pi} \|f\|_{L^1} r_0^{-3} \leq \|f_{in}\|_{L^\infty} + \frac{3}{4\pi} \|f_{in}\|_{L^1} r_0^{-3}, \quad \text{on } \partial_p Q.$$

We now apply Lemma 5.1 to $f(v, t)$: first note that (5.2) is a consequence of Lemma 5.3. Thanks to the uniform integrability of $f^{3/2}$ (5.10), Lemma 5.1 (for some $\gamma < 1/2$) yields

$$\sup_{t \in [0, T]} \|f(t)\|_{L^6(B_{r_0})} \leq \max\{r_0^{-5\gamma} \cdot \|f_{in}\|_{L^1(\mathbb{R}^3)}, \|f_{in}\|_{L^p_{\text{weak}}}\}, \quad \text{for some } p > 6.$$

It follows that $\|f(\cdot, t)\|_{L^6(\mathbb{R}^3)}$ and $\|\nabla a[f(\cdot, t)]\|_{L^\infty(\mathbb{R}^3)}$ are bounded for $t \leq T$. In particular,

$$\|f \nabla a[f]\|_{L^2(\mathbb{R}^3)} < \infty.$$

Applying (2.5) from Theorem 2.3 we arrive at

$$\|f\|_{L^\infty(Q)} \leq \|f\|_{L^\infty(\partial_p Q)} + C\|f \nabla a[f]\|_{L^2(Q)} < \infty,$$

which proves the lemma. \square

Proof of Theorem 1.1. Lemma 5.4 guarantees that if (5.10) holds, then any solution to (1.4) and (1.7) stays uniformly bounded in time.

6. MASS COMPARISON AND PROOF OF THEOREM 1.2

In this section we apply the ideas from previous sections to construct global solutions (in the radial, monotone case) for equation (1.7), namely

$$\partial_t f = a[f] \Delta f + f^2.$$

In view of Lemma 5.4, the fact that $T_0 = \infty$ in Theorem 1.1 results from a bound of any $L^p(\mathbb{R}^3)$ -norm of f , with $p > 3/2$. For (1.7) the bound of any $L^p(\mathbb{R}^3)$ -norm of f , with $p > 3/2$ will be proven by a barrier argument done at the level of the *mass function* of $f(v, t)$, which is defined by

$$M_f(r, t) = \int_{B_r} f(v, t) dv, \quad (r, t) \in \mathbb{R}_+ \times (0, T_0).$$

Depending on which problem f solves, the associated function $M_f(r, t)$ solves a one-dimensional parabolic equation with diffusivity given by $A^*[f]$ or $a[f]$.

Proposition 6.1. *Let f be a solution of (1.4) (resp. (1.7)) in $\mathbb{R}^3 \times [0, T_0]$, then $M(r, t)$ solves*

$$\partial_t M_f = A^* \partial_{rr} M_f + \frac{2}{r} \left(\frac{M_f}{8\pi r} - A^* \right) \partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0, T_0) \quad (6.1)$$

$$\left(\text{resp. } \partial_t M_f = a \partial_{rr} M_f + \frac{2}{r} \left(\frac{M_f}{8\pi r} - a \right) \partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0, T_0) \right). \quad (6.2)$$

Proof. We briefly show how to obtain (6.2); for (6.1) calculations are identical. Using the divergence theorem and the divergence expression in (1.7) we get

$$\partial_t M_f = \int_{\partial B_r} (a[f] \nabla f - f \nabla a[f], n) d\sigma = 4\pi r^2 (a[f] \partial_r f - f \partial_r a[f]).$$

Furthermore, straightforward differentiation yields the formulas

$$4\pi r^2 \partial_r f = r^2 \partial_r (r^{-2} \partial_r M_f), \quad \partial_r a[f] = -(4\pi r^2)^{-1} M_f.$$

Substituting these in the expression for $\partial_t M_f$ above we get

$$\partial_t M_f = a[f] r^2 \partial_r \left(\frac{1}{r^2} \partial_r M_f \right) + \frac{1}{4\pi r^2} M_f \partial_r M_f.$$

Expansion and rearrangement of the terms result in:

$$\begin{aligned} \partial_t M_f &= a \left(-\frac{2}{r} \partial_r M_f + \partial_{rr} M_f \right) + \frac{M_f}{4\pi r^2} \partial_r M_f \\ &= a \partial_{rr} M_f + \frac{2}{r} \left(\frac{M_f}{8\pi r} - a \right) \partial_r M_f, \end{aligned}$$

and the thesis follows. \square

Define the linear parabolic operator L in $\mathbb{R}_+ \times (0, T)$ as

$$Lh := \partial_t h - a \partial_{rr} h - \frac{2}{r} \left(\frac{M_f}{8\pi r} - a[f] \right) \partial_r h.$$

The above proposition simply says that $LM_f = 0$ in $\mathbb{R}_+ \times (0, T)$. The next proposition identifies suitable supersolutions for L .

Proposition 6.2. *If $m \in [0, 2]$ and $h(r, t) = r^m$ then $Lh \geq 0$ in $\mathbb{R}_+ \times (0, T)$.*

Proof. By direct computation we see that

$$Lh = -mr^{m-2} \left[(m-1)a + 2 \left(\frac{M_f}{8\pi r} - a[f] \right) \right].$$

On the other hand,

$$a[f](r) = \frac{1}{4\pi r} \int_{B_r} f \, dv + \int_{B_r^c} \frac{f}{4\pi|v|} \, dv \geq \frac{M_f}{4\pi r},$$

which guarantees that $\frac{1}{2}a[f](r) \geq \frac{M_f}{8\pi r}$. Thus,

$$\begin{aligned} Lh &= mr^{m-2} \left[(1-m)a[f] + 2 \left(a[f] - \frac{M_f}{8\pi r} \right) \right], \\ &\geq mr^{m-2} ((2-m)a[f]) \geq 0. \end{aligned}$$

The last inequality being true for $m \leq 2$. \square

Proof of Theorem 1.2. Since $f_{\text{in}} \in L^p$ for some $p > 3/2$, there is some $\alpha \in (0, 1)$ and some $C_0 > 0$ such that

$$M_{f_{\text{in}}}(r, 0) = \int_{B_r} f_{\text{in}} \, dv \leq C_0 r^{1+\alpha}.$$

Moreover, since $f(\cdot, t)$ has total mass 1 for every $t > 0$, we also have

$$M_f(r, t) \leq 1, \quad \forall r > 0, t \in (0, T).$$

Proposition 6.2 says that $h = Cr^{1+\alpha}$ is a supersolution of the parabolic equation solved by M_f in $\mathbb{R}_+ \times (0, T)$. Then choosing $C := \max\{C_0, 1\}$ comparison principle yields

$$M_f(r, t) \leq h(r) = Cr^{1+\alpha} \quad \text{for } r \in (0, 1), t \in (0, T). \quad (6.3)$$

Since $f(v, t)$ is readily symmetric and decreasing, bound (6.3) implies that $f(|v|, t) \leq \frac{3C}{4\pi} \frac{1}{|v|^{2-\alpha}}$ for $v \in B_1$; hence there is some $p' > 3/2$ and some $C_{p'} > 0$ such that

$$\|f(\cdot, t)\|_{L^{p'}(B_1)} \leq C_{p'}, \quad \forall t \in (0, T).$$

Then Lemma 5.1 says that $f(v, t)$ is bounded in $\mathbb{R}^3 \times (0, T_0)$. By Lemma 5.4, it follows $T_0 = +\infty$ and we have a global in time smooth solution. \square

As a corollary of Theorem 1.2 and Proposition 6.1 we give another criterium under which blow-up for the classical Landau equation is ruled out:

Corollary 6.3. *Suppose that for all $t \in [0, T_0]$ there is some $r_0 > 0$ and $0 < \lambda < 8\pi$ such that*

$$M_f(r, t) \leq \lambda r A^*(r, t) \quad \forall \quad r < r_0.$$

Then any solution to (1.4) is bounded for any $t > 0$.

7. APPENDIX

Proof of Proposition 3.1. The radial symmetry of any solution f to (2.1) follows by the uniqueness property of (2.1) and by the fact that $Q(g, f)$ commutes with rotations, as shown below. We first rewrite the collision operator as

$$Q(g, f) = \operatorname{div}(A[g]\nabla f - f\nabla a[g]) - (1 - \alpha)fg = a[g]\Delta f - \operatorname{div}(\tilde{A}[g]\nabla f) + \alpha fg,$$

with $\tilde{A}[g]\nabla f := \int \frac{g(|v-y|)}{|y|^3} \langle \nabla f(v), y \rangle y \, dy$.

Let \mathbb{T} be a rotation operator. Since g is radially symmetric, so is $a[g]$. Hence

$$a[g]\Delta(f \circ \mathbb{T}) = a[g \circ \mathbb{T}]\Delta(f \circ \mathbb{T}) = (a[g] \circ \mathbb{T})(\Delta f \circ \mathbb{T}) = (a[g]\Delta f) \circ \mathbb{T},$$

taking into account that the Laplacian operator commutes with rotations. Moreover

$$\begin{aligned} \operatorname{div}(\tilde{A}[g]\nabla f(\mathbb{T}v)) &= \operatorname{div}\left(\int \frac{g(|v-y|)}{|y|^3} \langle \nabla f(\mathbb{T}v), y \rangle y \, dy\right) \\ &= \operatorname{div}\left(\int \frac{g(|v-y|)}{|y|^3} \langle \mathbb{T}^* \nabla_z f(z)|_{z=\mathbb{T}v}, y \rangle y \, dy\right) \\ &= \operatorname{div}\left(\int \frac{g(|\mathbb{T}(v-y)|)}{|y|^3} \langle \nabla_z f(z)|_{z=\mathbb{T}v}, \mathbb{T}y \rangle \mathbb{T}^* \mathbb{T}y \, dy\right) \\ &= \operatorname{div}\left(\mathbb{T}^* \underbrace{\int \frac{g(|\mathbb{T}v-y|)}{|y|^3} \langle \nabla_z f(z)|_{z=\mathbb{T}v}, y \rangle y \, dy}_{=: V(\mathbb{T}v)}\right) \\ &= \operatorname{Tr}(\mathbb{T}^* \operatorname{Jac}(V)|_{z=\mathbb{T}v} \mathbb{T}) + \underbrace{\nabla(\operatorname{Tr}(\mathbb{T}^*))}_{=0} \cdot V(\mathbb{T}v) \\ &= \operatorname{Tr}(\mathbb{T} \mathbb{T}^* \operatorname{Jac}(V)|_{z=\mathbb{T}v}) \\ &= \operatorname{Tr}(\operatorname{Id} \operatorname{Jac}(V)|_{z=\mathbb{T}v}) \\ &= \operatorname{div}\left(\int \frac{g(|z-y|)}{|y|^3} \langle \nabla_z f(z), y \rangle y \, dy\right) \circ \mathbb{T}. \end{aligned}$$

Hence $Q(g, f(\mathbb{T}v)) = Q(g, f) \circ \mathbb{T}$.

Now we rewrite the linear equation (2.1) in spherical coordinates:

$$\partial_t f = A^* \partial_{rr} f + \frac{a-A^*}{r} \partial_r f + fg, \quad (7.1)$$

with $A^*[g](v) := (A[g](v)\hat{v}, \hat{v})$, $\hat{v} := \frac{v}{|v|}$ and differentiate (7.1) with respect to r . The function $w := \partial_r f$ satisfies the following inequality:

$$\partial_t w \leq A^* \partial_{rr} w + \frac{a-A^*}{r} \partial_r w + wg + \partial_r A^* \partial_r w + \partial_r \left(\frac{a-A^*}{r}\right) w.$$

If $w(\cdot, 0) \leq 0$ it follows from maximum principle that $w(\cdot, t) \leq 0$ for all $t \geq 0$. In other words, the (negative) sign of $\partial_r f$ is preserved in time. \square

Proof of Proposition 3.2. The identity (3.3) is a classical and a proof can be found in [11][Section 9.7]. To prove (3.2), let $v \in \mathbb{R}^3$ non-zero, $r := |v|$, then

$$(A[g](v)\hat{v}, \hat{v}) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{1}{|v-w|} g(w) \left(\left(\mathbb{I} - \frac{v-w}{|v-w|} \otimes \frac{v-w}{|v-w|} \right) \hat{v}, \hat{v} \right) dw.$$

Note that

$$\left(\left(\mathbb{I} - \frac{v-w}{|v-w|} \otimes \frac{v-w}{|v-w|} \right) \hat{v}, \hat{v} \right) = 1 - \cos(\hat{\theta}(w))^2,$$

where $\hat{\theta}$ denotes the angle between $w - v$ and v . Consider, for $0 \leq t, r$, the function

$$I(r, t) := \int_{\partial B_t} \frac{1 - \cos(\hat{\theta})^2}{|v-w|} dw.$$

The function $I(r, t)$ encodes all the information about A^* . In particular, integration in spherical coordinates yields the expression

$$A^*[h](v) = \frac{1}{8\pi} \int_0^\infty f(t) I(|v|, t) dt.$$

As it turns out, $I(r, t)$ has rather different behavior according to whether $r < t$ or not. By averaging in the v variable, it is not hard to see that

$$I(r, t) = \frac{t^2}{r^4} I(t, r), \quad \forall r < t.$$

Accordingly, we focus on $I(r, t)$ when $r > t$. To do so, denote by θ the angle between w and v and observe that

$$1 - \cos(\hat{\theta})^2 = \sin(\hat{\theta})^2 = \frac{t^2 - t^2 \cos(\theta)^2}{|v-w|^2} = \frac{t^2 - w_1^2}{|v-w|^2},$$

where $w_1 = (w, \hat{v})$. Thus,

$$\begin{aligned} I(r, t) &= \int_{\partial B_t} \frac{t^2 - w_1^2}{|v-w|^3} dw \\ &= \int_{\partial B_t} \frac{t^2 - w_1^2}{(t^2 - w_1^2 + (r - w_1)^2)^{3/2}} dw \\ &= \int_{\partial B_t} \frac{t^2 - w_1^2}{(t^2 - 2rw_1 + r^2)^{3/2}} dw \\ &= \int_{\partial B_1} \frac{t^2(1 - z_1^2)}{t^3(1 - 2(\frac{r}{t})z_1 + (\frac{r}{t})^2)^{3/2}} t^2 dz \\ &= \int_{\partial B_1} \frac{1 - z_1^2}{(1 - 2(\frac{r}{t})z_1 + (\frac{r}{t})^2)^{3/2}} t dz. \end{aligned}$$

This surface integral can be written entirely as an integral in terms of the variable $z_1 \in (-1, 1)$,

$$I(r, t) = 2\pi t \int_{-1}^1 \frac{1 - z_1^2}{(1 - 2(\frac{r}{t})z_1 + (\frac{r}{t})^2)^{3/2}} dz_1.$$

For brevity, set for now $s = r/t$, then

$$\begin{aligned} \int_{-1}^1 \frac{1 - z_1^2}{(1 - 2sz_1 + s^2)^{3/2}} dz_1 &= \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3\sqrt{s^2 - 2s + 1}} - \frac{-2s^4 - 2s^3 - 2s - 2}{3s^3\sqrt{s^2 + 2s + 1}} \\ &= \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3(s - 1)} - \frac{-2s^4 - 2s^3 - 2s - 2}{3s^3(s + 1)} \\ &= \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3(s - 1)} + \frac{2s^4 + 2s^3 + 2s + 2}{3s^3(s + 1)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3(s - 1)} + \frac{2s^4 + 2s^3 + 2s + 2}{3s^3(s + 1)} &= \frac{2}{3s^3} \left(\frac{-s^4 + s^3 + s - 1}{s - 1} + \frac{s^4 + s^3 + s + 1}{s + 1} \right) \\ &= \frac{2}{3s^3} \frac{(-s^4 + s^3 + s - 1)(s + 1) + (s^4 + s^3 + s + 1)(s - 1)}{s^2 - 1} \\ &= \frac{2}{3s^3} \frac{2s^2 - 2}{s^2 - 1} = \frac{4}{3s^3}. \end{aligned}$$

Then, since $s = r/t$, we conclude that

$$\begin{aligned} I(r, t) &= 8\pi \frac{t^4}{3r^3}, \quad \text{for } t < r, \\ I(r, t) &= 8\pi \frac{1}{3t}, \quad \text{for } t > r. \end{aligned}$$

Going back to $A^*[h]$, the above leads to

$$\begin{aligned} A^*[h](v) &= \int_0^r h(t) I(r, t) dt + \int_r^\infty h(t) I(r, t) dt \\ &= \frac{1}{3r^3} \int_0^r h(t) t^4 dt + \frac{1}{3} \int_r^\infty h(t) t dt. \end{aligned}$$

□

Proof of Lemma 5.3. This argument is inspired by the one in Section 2.6 in [9]. For β, R, r (with $0 < r < R$, $0 < \beta$) consider the function

$$\Phi(v, t) := e^{-\beta t} (|v| - R)^2 (|v| - r)^2.$$

Since Φ is a $C^{1,1}$ function with compact support, it holds

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} f(v, t) \Phi(v) dv &= - \int_{\mathbb{R}^3} (a \nabla f - f \nabla a, \nabla \Phi) dv \\ &= \int_{\mathbb{R}^3} f \operatorname{div}(a \nabla \Phi) dv + \int_{\mathbb{R}^3} f (\nabla a, \nabla \Phi) dv. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{div}(a \nabla \Phi) + (\nabla a, \nabla \Phi) &= a \Delta \Phi + 2(\nabla a, \nabla \Phi) \\ &= a \Phi'' + \frac{2}{|v|} (a + |v| a') \Phi' \\ &= a \Phi'' + \frac{2}{|v|} \Phi' \int_{|v|}^{+\infty} s f(s, t) ds. \end{aligned}$$

It holds:

$$\begin{aligned}\Phi'(s) &= 2(R-s)(s-r)(-(s-r)+R-s) = 2(R-s)(s-r)(R+r-2s), \\ \Phi''(s) &= 2(R-s)(r+R-2s) - 2(s-r)(r+R-2s) - 4(R-s)(s-r), \\ \Phi'(r) &= \Phi'(R) = 0, \quad \Phi''(r) = \Phi''(R) = 2(R-r)^2, \\ |\Phi''|, |\Phi'| &\leq C_{\delta,r,R}\Phi, \quad |v| \in ((1+\delta)r, (1-\delta)R).\end{aligned}$$

Hence in a small neighborhood of $|v| = R$ and $|v| = r$ one can show that $\frac{d}{dt} \int_{\mathbb{R}^3} f(v, t) \Phi(v) dv \geq 0$; more precisely it holds

$$\operatorname{div}(a \nabla \Phi) + (\nabla a, \nabla \Phi) \geq 0 \text{ in } B_R \setminus B_{(1-\delta)R} \cup B_{(1+\delta)r} \setminus B_r.$$

Since $a[g](v) \leq \frac{\|g\|_{L^1(\mathbb{R}^3)}}{|v|}$, it follows

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^3} f(v, t) \Phi(v) dv &\geq -C_{\delta,r,R} \frac{\|g\|_{L^1(\mathbb{R}^3)}}{r} \int_{B_{(1-\delta)R} \setminus B_{(1+\delta)r}} f(v, t) \Phi(v) dv \\ &\geq -\frac{\|g\|_{L^1(\mathbb{R}^3)}}{r} C_{\delta,r,R} \int_{\mathbb{R}^3} f(v, t) \Phi(v) dv.\end{aligned}$$

This above differential inequality implies

$$\int_{\mathbb{R}^3} f(v, t) \Phi(v) dv \geq e^{-\beta t} \int_{\mathbb{R}^3} f_{\text{in}} \Phi(v) dv, \quad \forall t < T,$$

where $\beta = C_{r,R,\alpha} \|g\|_{L^1}$. Finally, since

$$\Phi(v) \leq \frac{1}{4}(R-r)^2 \text{ in } B_R \setminus B_r, \quad \Phi(v) \geq \frac{R^2 r^2}{4},$$

we conclude that

$$\int_{B_R \setminus B_r} f(v, t) dv \geq \frac{R^2 r^2}{(R-r)^4} e^{-\beta t} \int_{B_{R/2} \setminus B_{2r}} f_{\text{in}}(v) \Phi(v) dv, \quad \forall t < T.$$

□

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